Geometric Series and Infinite Games

Introduction

One of the most interesting questions often asked is "what is infinity?" This question has not only perplexed learners, but also mathematicians and scientists who are still trying to make sense of this fascinating concept. The number of grains of sand on a beach, the amount of atoms in our bodies or even the sum of all the stars in the observable universe are finite. In fact, everything that we see around us appears to be finite. The question is therefore, does infinity really exist? Well, mathematics has given us a better understanding of the concept of infinity and its rather peculiar properties.

Aim of the Workshop

The aim of this workshop is to introduce students to the concept of infinity and its applications in counting problems and infinite sums. In particular, the workshop will outline some of the unusual behaviours of infinity as seen in the Menger Sponge and the Hilbert's Hotel paradoxes. Different types of infinite sums will also be discussed, including sums which converge to a finite number, sums which diverge to infinity, and sums that do neither (e.g. Grandi's Series).

Learning Outcomes

By the end of this workshop students should be able to:

- Recognise the Sigma notation and understand how it is used
- Explain, in their own words, what is meant by infinity
- Describe some of the unusual properties of infinity (Hilbert Hotel, Menger Sponge, etc.)

Materials and Resources

Optional: Hilbert's hotel video clip
https://www.youtube.com/watch?v=faQBrAQ87l4

Keywords

Infinity
An abstract concept used to describe something that is unbounded or greater than any known quantity.

Geometric Sequence
An ordered list of numbers in which each term is found by multiplying the previous term by a constant.

Geometric Series
A geometric series is the sum of the numbers in a geometric sequence.

Geometric Series & Infinite Games: Workshop Outline

SUGGESTED TIME (TOTAL MINS)	ACTIVITY	DESCRIPTION
5 mins (00:05)	**Introduction to the Concept of Infinity**	– Define what is meant by Infinity (see keywords) – Introduce students to the Menger Sponge (**see Appendix– Note 1**)
15 mins (00:20)	**Geometric Series**	– Ask students "What is a sequence?" – Once student feedback is discussed, explain that a sequence is an ordered list of numbers and that the sum of such sequences is called a 'series' – Define what is meant by both a Geometric Sequence and Geometric Series (see keywords)
15–20 mins (00:40)	**Sigma Notation**	– Introduce students to the Sigma Notation Σ – Explain to the students that sigma means "sum of" and is used to find the sum of a sequence – **Activity Sheet 1:** Divide students into pairs and ask them to complete activity sheet 1 (see **Appendix– Note 2**)
5 mins (00:45)	**Discussion on Activity 1**	– Once Activity 1 is complete, ask students: • "If we *decrease* the number on top of Sigma, will our result become larger or smaller than before?" • "What if we *increase* the number on below Sigma, will our result become larger or smaller than before?" • (it is important to note the different types of sequences) • *"What about Infinity?"*
15 mins (01:00)	**Hilbert's Hotel**	– Introduce students to the Hilbert's Hotel paradox (see **Appendix – Note 3**) – Divide students into groups and ask them "If Hilbert's Hotel is fully booked, is it possible to accommodate another guest? (see **Appendix– Note 4**) – Ask students "A bus arrives containing an infinite number of guests who wish to stay at the hotel. How can we accommodate them if all rooms are fully booked?" (see **Appendix – Note 5**) – Whole class discussion on possible solutions. – Optional: Play a brief video clip on Hilbert's Hotel which explains the solutions to the above situations (link above in additional resources)

SUGGESTED TIME (TOTAL MINS)	ACTIVITY	DESCRIPTION
10 mins (01:10)	**Achilles and the Tortoise (Zeno's Paradox)**	– **Activity Sheet 2** In pairs, students complete Activity Sheet 2 – Whole class discussion on the activity – Explain the idea behind Zeno's paradox (**see Appendix– Note 6**)
10 mins (01:20)	**Grandi's Series**	– Explain the Grandi's Series to the students $1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + \ldots\ldots$ (see **Appendix– Note 7**) – **Activity sheet 3 –** In pairs, students complete Activity Sheet 3

Geometric Series and Infinite Games Workshop Appendix

Note 1: Menger Sponge

Menger Sponge is a theoretical shape that has infinite surface area and no volume. It is also known as a fractal curve meaning it exhibits the same repeating pattern at every scale.

The Menger sponge begins with a solid cube. This is divided into 27 smaller cubes and the centre from each of these cube faces is removed.

This process is repeated with the remaining cubes, leading to similar shape to that shown in figure 1. In doing so, the volume reduces and surface area increases each time. Given that this process can extend to infinity, Menger Sponge will thus have infinite surface area and zero volume.

Figure 1: Diagram of the Menger Sponge

Note 2: Activity Sheet 1 Question 3 Solutions

(a). $\sum_{n=1}^{\infty} 2^n$

(b). $\sum_{n=0}^{\infty} (5)3^n$

(c). $\sum_{n=0}^{\infty} \frac{81}{3^n}$

(d). $\sum_{n=0}^{\infty} (-1)^n$

Note 3: An Introduction to Hilbert's Hotel

Hilbert's Hotel is a mathematical paradox that is used to demonstrate some of the unusual properties of infinity. In a standard hotel, there are a finite number of rooms. Once each of the rooms has been assigned to a guest, the hotel is considered fully booked. However, in the case of Hilbert's hotel, there are an infinite number of rooms.

Note 4: Accommodating an Extra Guest in Hilbert's Hotel

If all the rooms in Hilbert's Hotel are booked out, it might appear that no more guests can be accommodated. Fortunately, however, a room can be provided for an additional guest by moving the guest staying in room 1 to room 2. The guest in room 2 then moves to room 3 and so on. Room 1 will thus be available for the new guest once everyone else has moved accordingly. Representing this mathematically, the guest in room **n** will be moved to room **n+1**. This demonstrates how it is possible to accommodate a new guest even if the hotel is already fully booked, something that could not happen in a hotel with a finite number of rooms.

Note 5: Accommodating an Infinite Number of Guests in Hilbert's Hotel

If a bus then arrives with an infinite number of guests, it is still possible to accommodate them in Hilbert's Hotel despite each room being occupied. This time, instead of moving each guest to the room beside them (i.e. n+1), we ask them to move to the room which is double their current one. In other words, the guest in room 2 moves to room 4, the guest in room 3 moves to room 6 and so on. This leaves the infinitely many odd numbered rooms free which can thus accommodate the infinite number of guests. Representing this mathematically, the guest in room **n** will be moved to room **2n**. This paradox is an interesting way to demonstrate the unusual properties of infinity.

Note 6: Zeno's Paradox

Greek philosopher Zeno designed a paradox to describe a way in which a tortoise could win a 1 Kilometre race against the legendary hero "Achilles".

The tortoise is given a head start of 500 metres. Once the race begins, Achilles would first have to first cover the distance to the point that the tortoise started. Meanwhile, the tortoise would have moved a little further from the 500 metre mark. Achilles would then have to cover that distance too, giving the tortoise time to move forward even more.

Whilst the gap between the two may reduce in size over time, Zeno pointed out that this process could go on infinitely long, given that the tortoise would be able to move forward each time Achilles is catching up. Thus Achilles could never win. This paradox led to the realisation that something finite could be divided an infinite number of times. (Where "…." means the pattern is recurring.)

$$1 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}.........$$

Note 7: Grandi's Series

The Grandi's Series, shown below, is an infinite series named after Italian mathematician Guido Grandis.

$$1-1+1-1+1-1+1-1+1-1........$$

By using parentheses, there are different ways of adding the Grandi's series, each of which produces a contradictory result. Hence it is also known as a divergent series, meaning it does not have a sum.

For example: $(1 - 1) + (1 - 1) + (1 - 1) + ... = 0 + 0 + 0 + ... = 0$.

On the other hand, the following parentheses give:

$1 + (-1 + 1) + (-1 + 1) + (-1 + 1) + ... = 1 + 0 + 0 + 0 + ... = 1$.

Thus, we can obtain both 0 and 1.

Sources and Additional Resources

http://wordplay.blogs.nytimes.com/2016/05/30/frenkel–cantor/

https://www.youtube.com/watch?v=faQBrAQ87l4 (Hilbert's hotel video clip)

http://mathandmultimedia.com/2014/05/26/grand–hotel–paradox/

http://www.mathsisfun.com/algebra/sigma–notation.html

http://platonicrealms.com/encyclopedia/zenos–paradox–of–the–tortoise–and–achilles

Geometric Series & Infinite Games: Activity Sheet 1

Geometric Series

1. **Can you find the next 2 terms in each of the following sequences?**

(a) 2, 4, 8, 16, ___, ___

(b) 5, 15, 45, 135, ___, ___

(c) 81, 27, 9, 3, ___, ___

(d) 1, –1, 1, –1, 1, ___, ___

2. **For each of the following, write down what you think the geometric sequence is (and solve it if you can!)**

$$\sum_{n=1}^{4} 3n$$ **(a) =**

$$\sum_{n=1}^{3} (2n+1)$$ **(b) =**

$$\sum_{n=2}^{5} n^2$$ **(c) =**

3. **Can you write the sequences in Question 1 using sigma notation?**

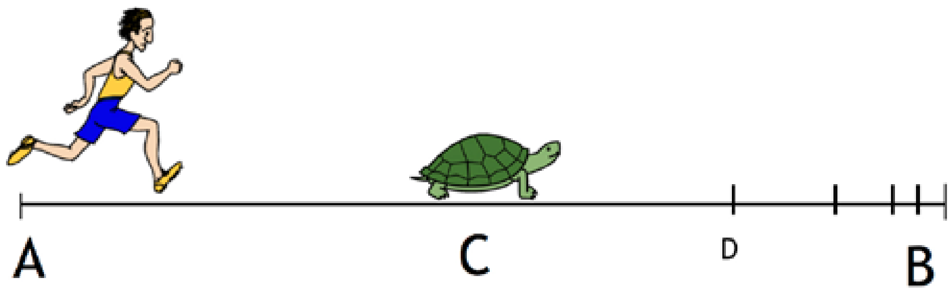
Geometric Series & Infinite Games: Activity Sheet 2

Achilles and the Tortoise

Achilles and his tortoise Hillary decide to have a race over 1 Kilometre. Given that Achilles runs twice as fast as Hillary, he decides to give her a head start of 500 metres from point C (the halfway mark).

The race begins and after Achilles arrives at C, Hillary has moved a certain distance ahead to a point D (midpoint between C and B).

Similarly, Achilles must then travel to point D. By this time, Hillary has moved onto point E (midpoint between D and B). Label this point.



1. **What is the distance from C to D? (In km)**

2. **What is the distance from D to E in? (In km)**

3. **Continuing in a similar pattern, how many "halfway points" will Achilles have to pass in order to reach B?**

Geometric Series & Infinite Games: Activity Sheet 2

4. Based on your previous answer, do you think Achilles will ever reach Hillary? Explain your reasoning.

5. What would happen if we were to write this as a geometric series?

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots\ldots=???$$

6. Looking back at Activity 1, how would we write this with Sigma notation?

$$\sum_{n=}$$

Geometric Series & Infinite Games: Activity Sheet 3

Grandi's Series

Below is the famous Grandi's series.

1-1+1-1+1-1+1-1+1-1........

1. Can you figure out the following "partial sums" of the Grandi's Series?

(a) $\sum_{n=0}^{2}(-1)^n$ =

(b) $\sum_{n=0}^{3}(-1)^n$ =

(c) $\sum_{n=0}^{5}(-1)^n$ =

(d) $\sum_{n=0}^{10}(-1)^n$ =

(e) $\sum_{n=0}^{15}(-1)^n$ =

(f) $\sum_{n=0}^{38}(-1)^n$ =

2. Can you see a pattern? Discuss with your partner.

(g) $\sum_{n=0}^{\infty}(-1)^n$ =